كلية الحاسبات والذكاء الإصطناعي

BS111
Probability and Statistics

Lecture 10

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2021

Ch 3.1: Discrete Prob. Distr.

- Discrete Uniform Distribution.
- Binomial and Multinomial Distributions.
- Negative Binomial Distributions.
- Geometric Distributions.
- Poisson Distribution.

Discrete Uniform Dist. (1/6)

Definition:

A random variable X has a discrete uniform distribution if each of the n values in its range, $x_1, x_2, \ldots, x_n$, has equal probability. Then

$$f(x_i) = \frac{1}{n}$$

Discrete Uniform Dist. (2/6)

Example1 (1/2):

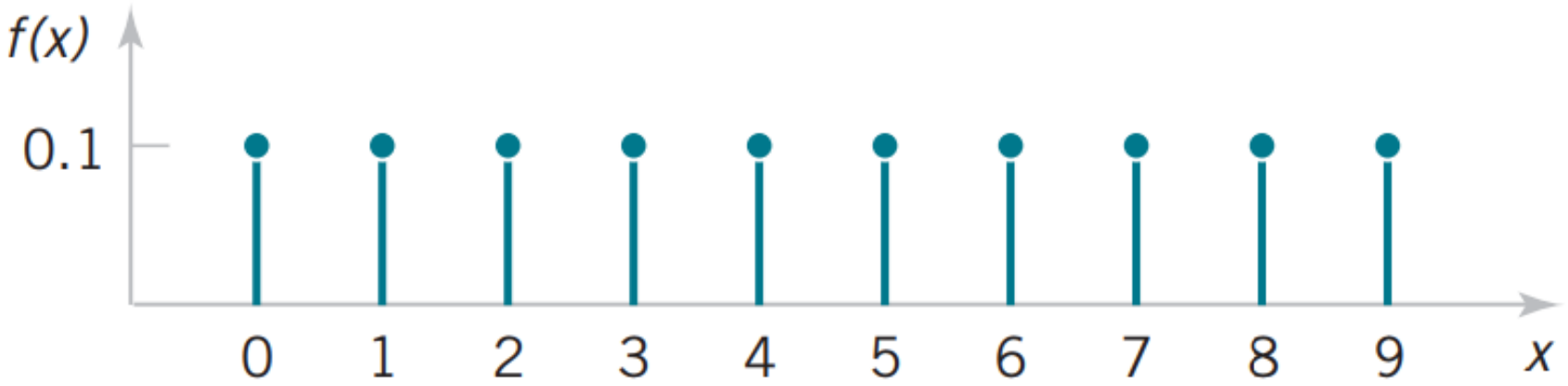
The first digit of a part's serial number is equally likely to be any one of the digits 0 through 9. If one part is selected randomly from a large batch and X is the first digit of the serial number, X has a discrete uniform distribution with probability 0.1 for each value in $R = \{0, 1, 2, \dots, 9\}$. That is,

$$f(x) = \frac{1}{10} = 0.1$$

Discrete Uniform Dist. (2/6)

Example1 (2/2):

Probability mass function for a discrete uniform random variable

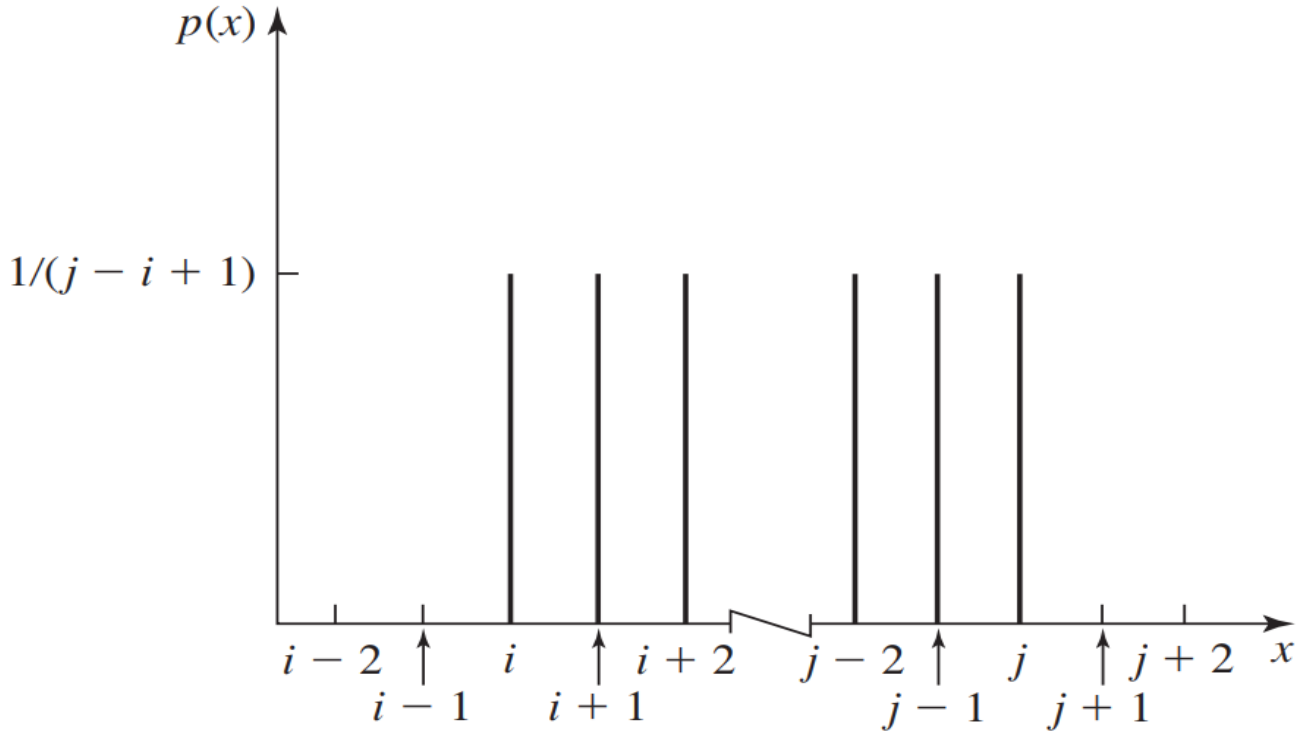


Discrete Uniform Dist. (3/6)

Discrete Uniform DU(i, j):

$X \sim \mathrm{DU}(i, j)$

Suppose that the range of the discrete random variable X equals the consecutive integers $i, i+1, \ldots, j$, for $i \leq j$



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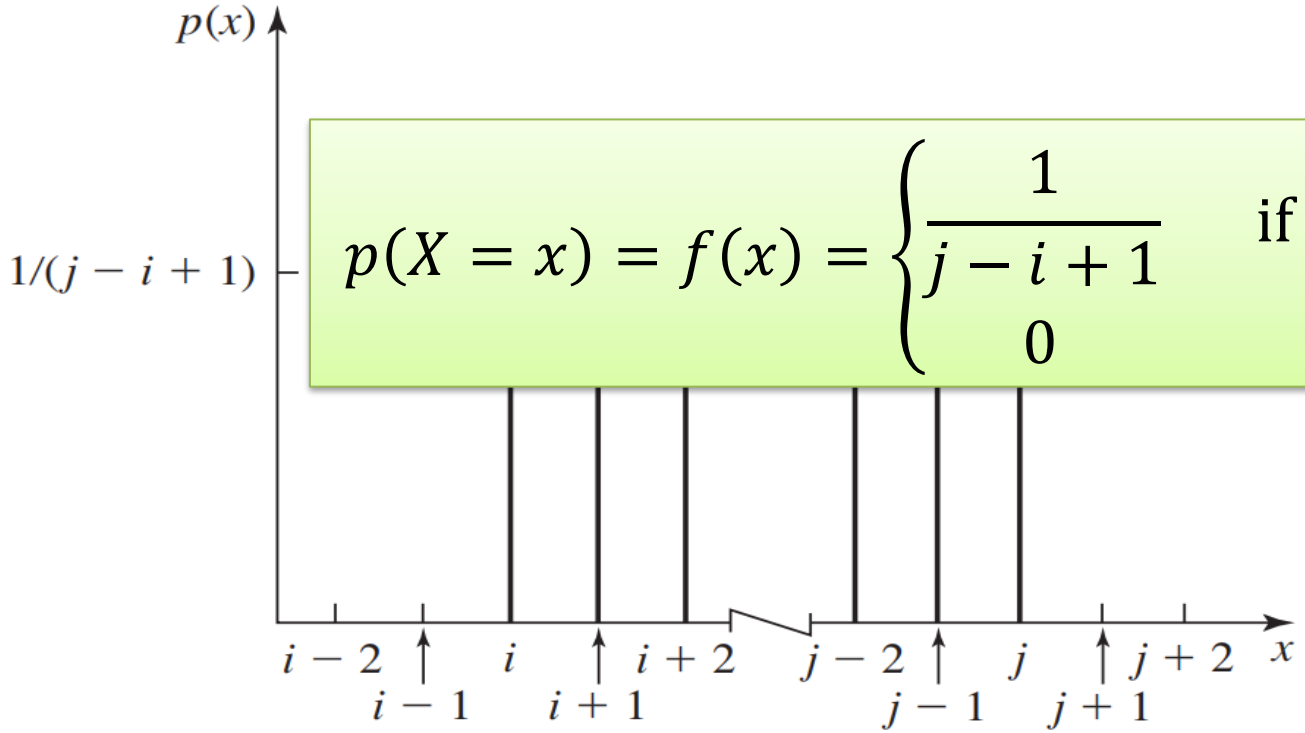
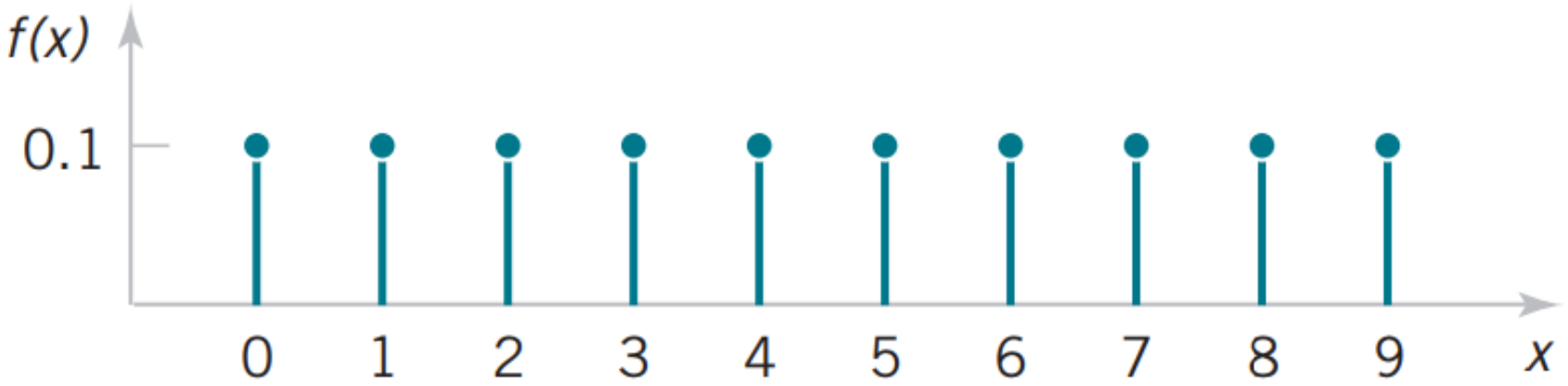


$$p(X = x) = f(x) = \begin{cases} \dfrac{1}{j - i + 1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

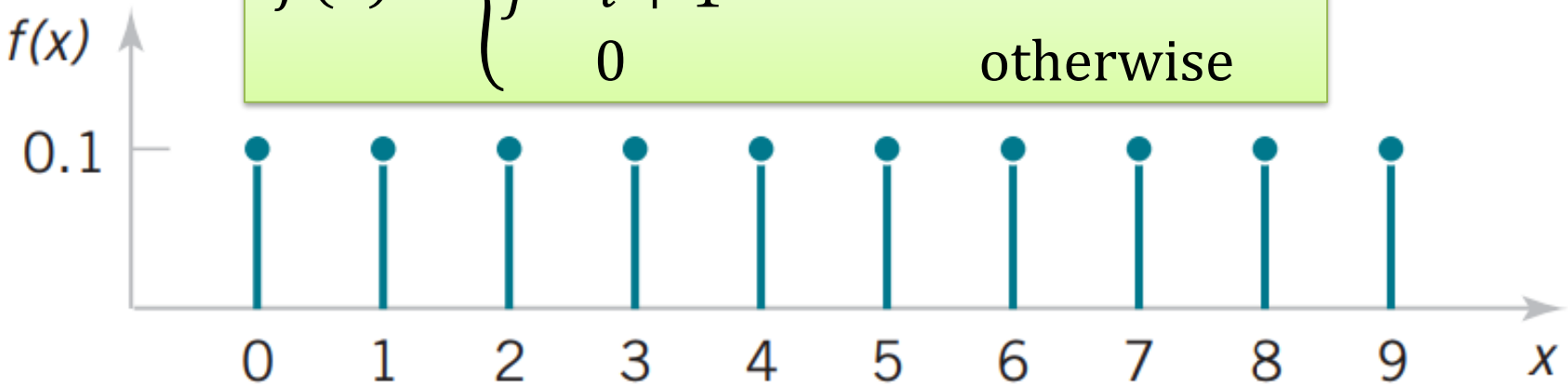


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{j-i+1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

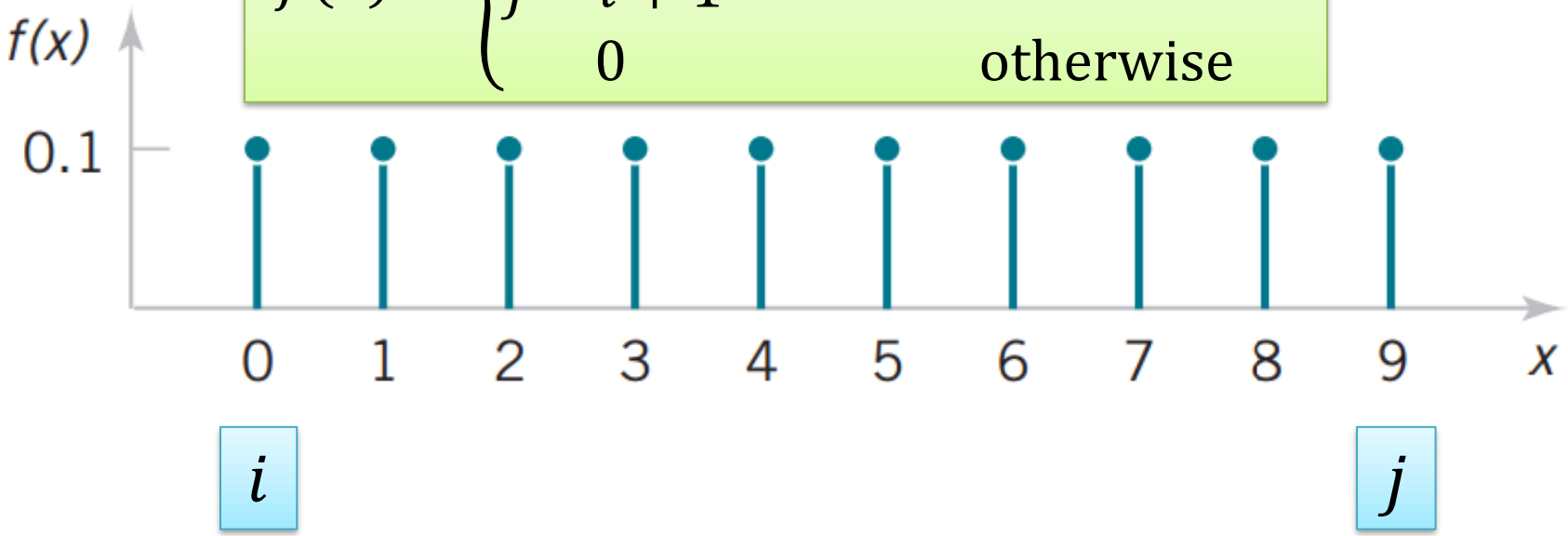


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{j-i+1} & \text{if } x \in \{i, i+1, \ldots, j\} \\ 0 & \text{otherwise} \end{cases}$$

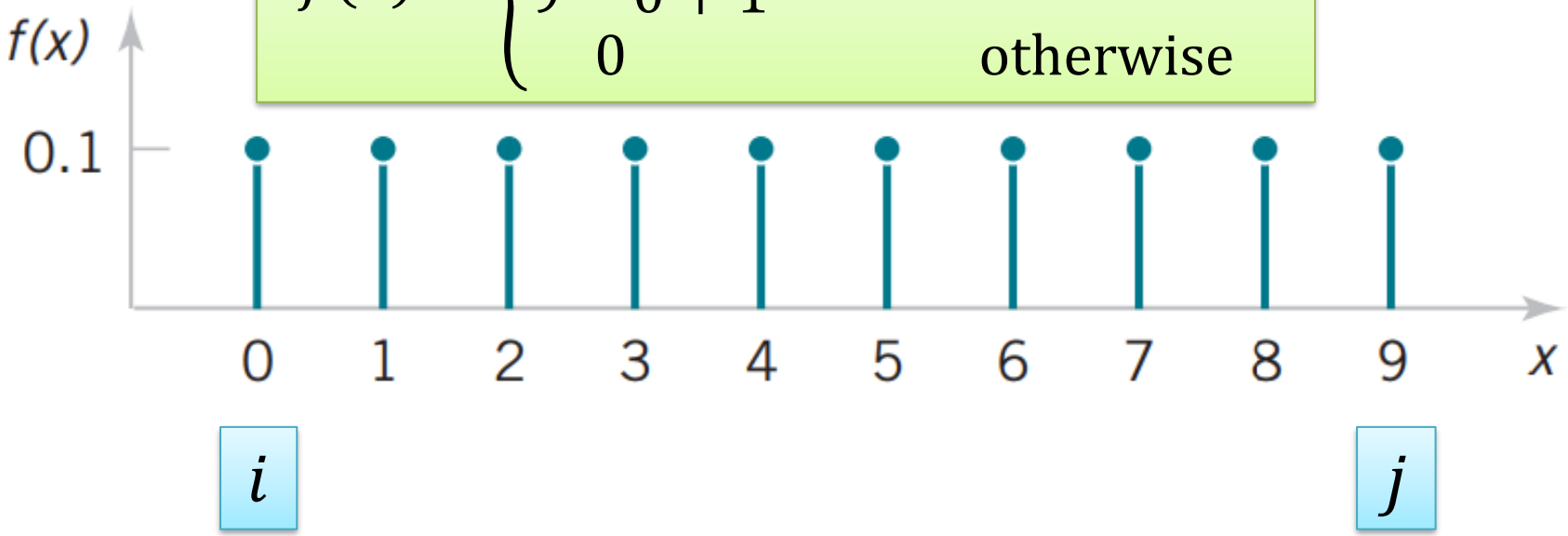


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} \dfrac{1}{9-0+1} & \text{if } x \in \{0, 1, \ldots, 9\} \\ 0 & \text{otherwise} \end{cases}$$

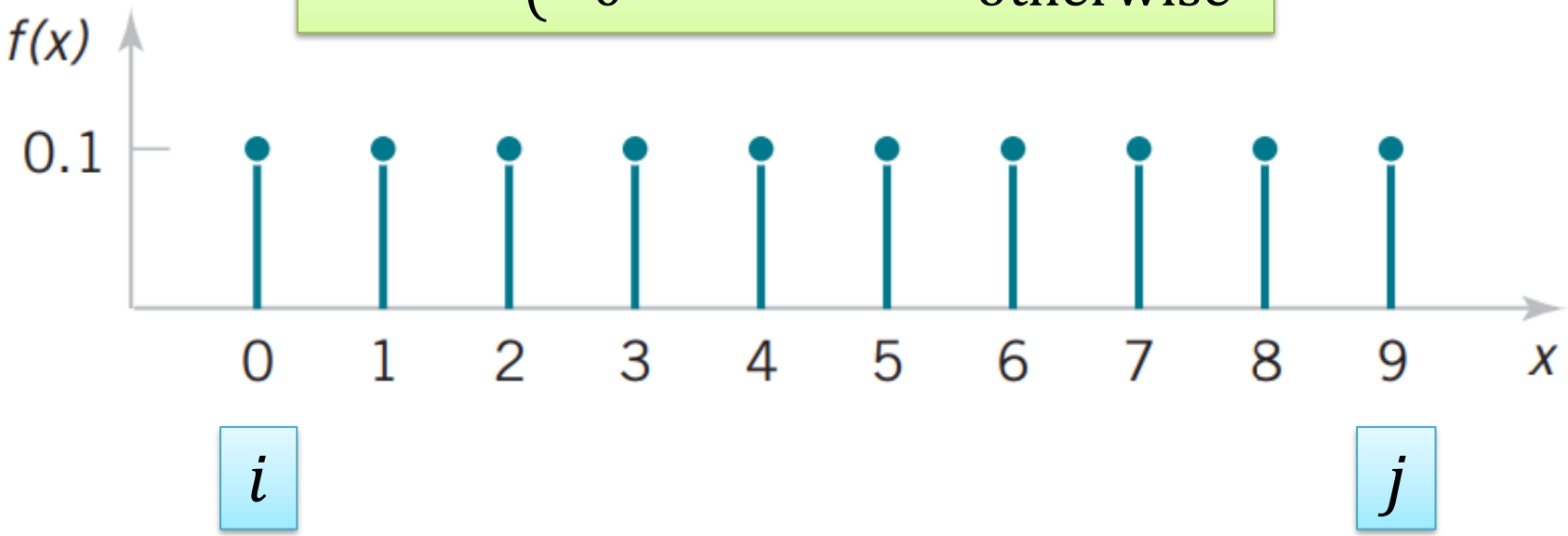


Discrete Uniform Dist. (4/6)

Recall Example1:

Probability mass function for a discrete uniform random variable

$$f(x) = \begin{cases} 1/10 & \text{if } x \in \{0, 1, \ldots, 9\} \\ 0 & \text{otherwise} \end{cases}$$



Discrete Uniform Dist. (5/6)

Mean and Variance:

Suppose that the range of the discrete random variable X equals the consecutive integers $i, i+1, \ldots, j$, for $i \leq j$

$$\mu = E(X) = \frac{j+i}{2}$$

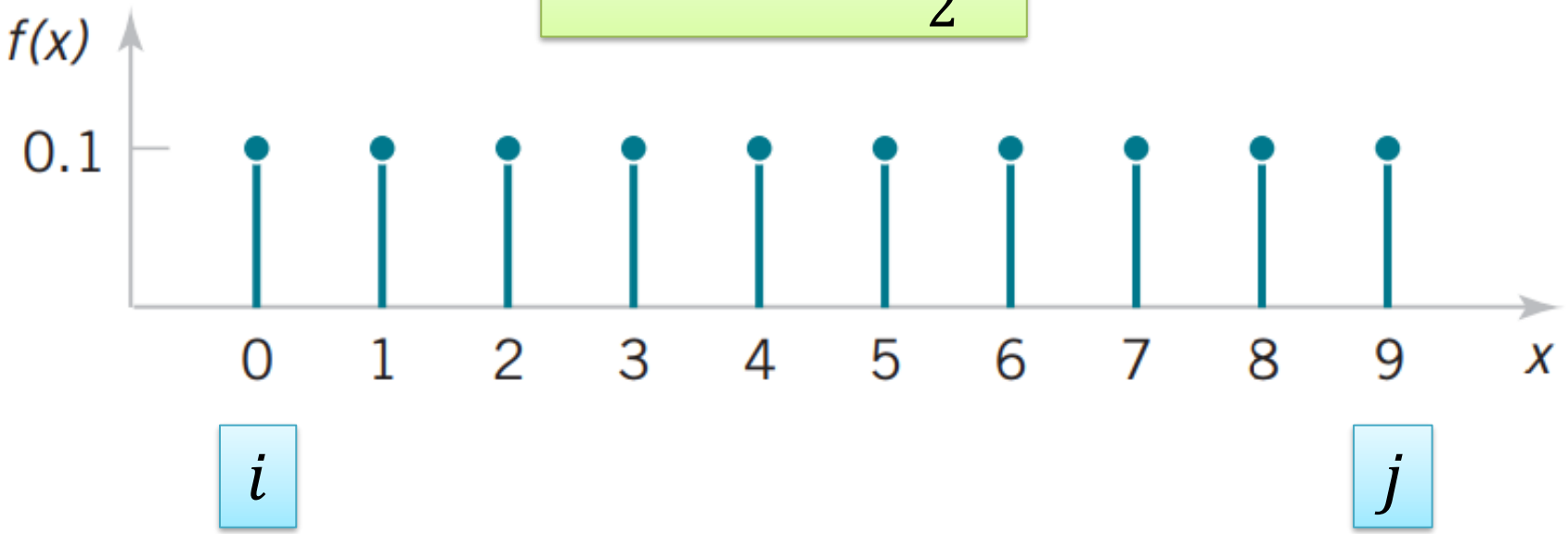
$$\sigma^2 = V(X) = \frac{(j-i+1)^2 - 1}{12}$$

Discrete Uniform Dist. (6/6)

Example2 (1/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\mu = E(X) = \frac{j+i}{2}$$

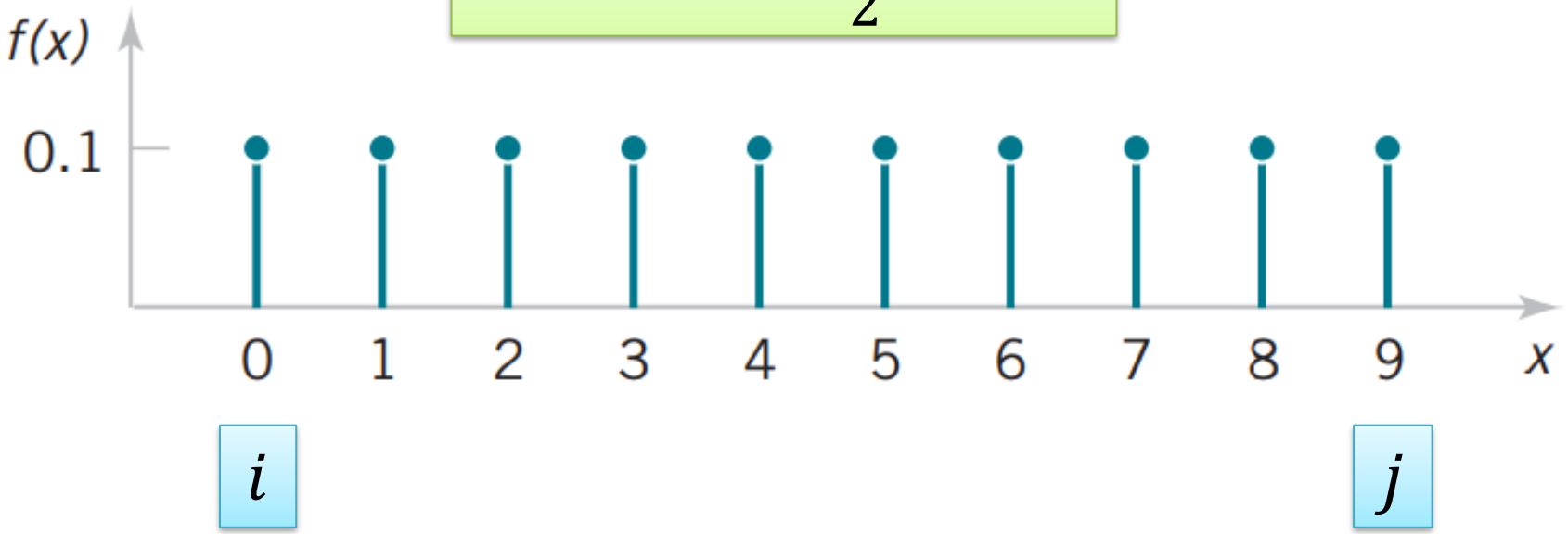


Discrete Uniform Dist. (6/6)

Example2 (2/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\mu = E(X) = \frac{9+0}{2} = 4.5$$

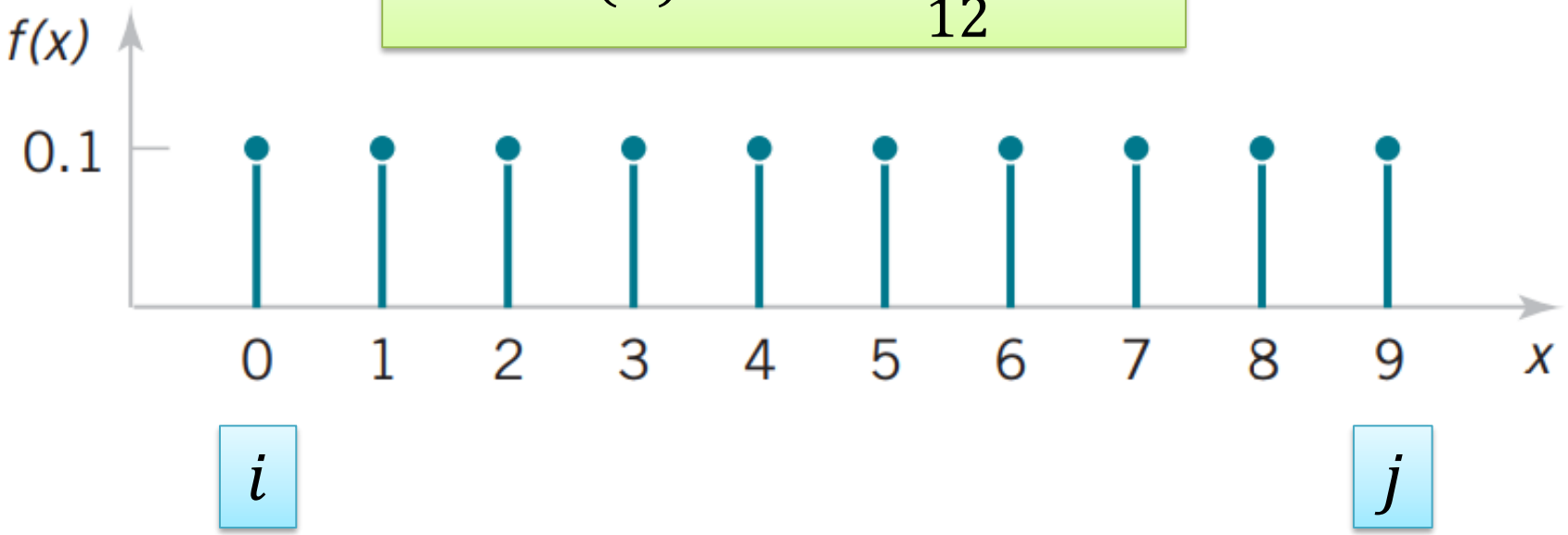


Discrete Uniform Dist. (6/6)

Example2 (3/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\sigma^2 = V(X) = \frac{(j-i+1)^2 - 1}{12}$$

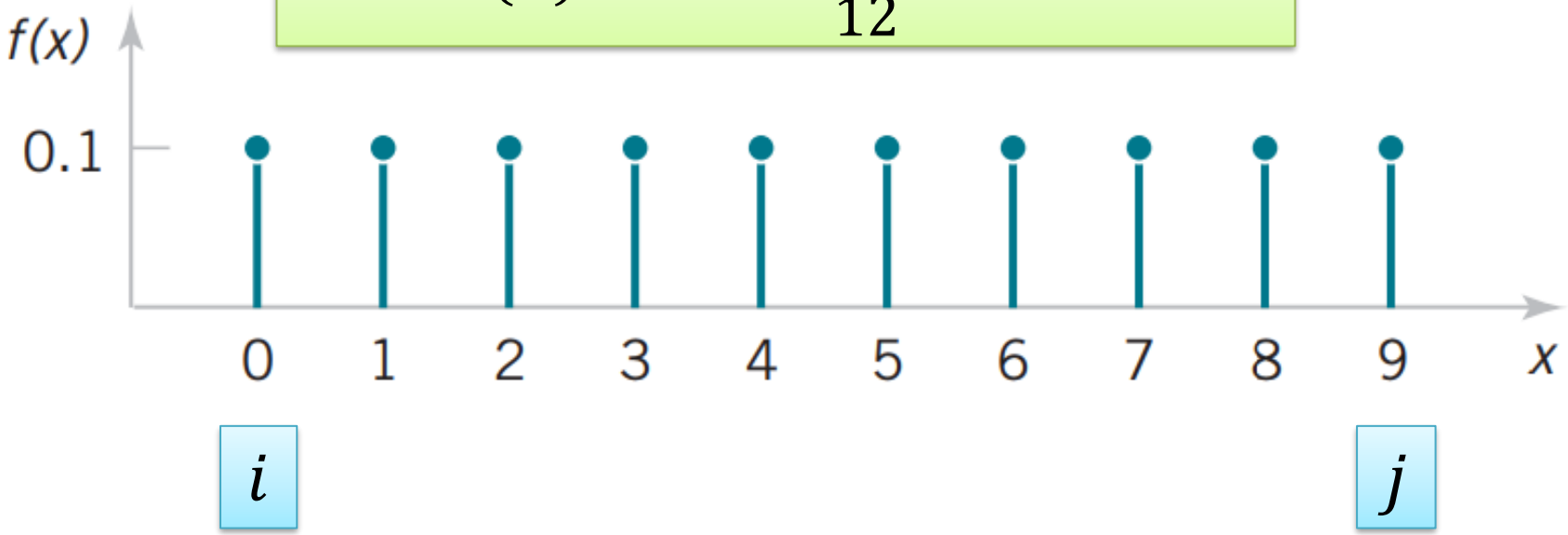


Discrete Uniform Dist. (6/6)

Example2 (4/4):

$f(x)$ is a probability mass function for a discrete uniform random variable X

$$\sigma^2 = V(X) = \frac{(9-0+1)^2-1}{12} = 8.25$$



Binomial Distribution (1/11)

The Bernoulli Process:

1. The experiment consists of ***repeated*** trials and each trial is called a ***Bernoulli trial***.
2. Each trial results in an outcome that may be classified as a ***success*** or a ***failure***.
3. The probability of success, denoted by $\boldsymbol{p}$, remains constant from trial to trial.
4. The repeated trials are ***independent***.

Binomial Distribution (2/11)

The Bernoulli Process: (Examples)

1. Flip a coin 10 times. Let X = number of heads obtained.
2. A multiple-choice test contains 10 questions, each with four choices, and you guess at each question. Let X = the number of questions answered correctly.
3. In the next 20 births at a hospital, let X = the number of female births.

Binomial Distribution (3/11)

Binomial Distribution $b(n,p)$:

$X \sim b(n,p)$ $X \sim \text{Bin}(n,p)$

A random experiment consists of n Bernoulli trials. The random variable X that equals the number of trials that result in a success is a *binomial random variable* with parameters $0 < p < 1$ and finite $n = 1, 2, \ldots$.

The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n$$

Binomial Distribution (4/11)

Example 1 (1/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

Binomial Distribution (4/11)

Example 1 (2/5):

Flip a coin 10 times, what is the probability that 3 heads occurs?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/11)

Example 1 (3/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that 3 heads occurs?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/11)

Example 1 (4/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that 3 heads occurs?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (4/11)

Example 1 (5/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that 3 heads occurs?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(3) = \binom{10}{3} (0.5)^3 (0.5)^{10-3}$$

$$= 120\ (0.5)^3 (0.5)^7$$

$$= 0.1171875$$

Binomial Distribution (5/11)

Example 2 (1/5):

Flip a coin 10 times, what is the probability that:

a) No head occurs.

b) At least 2 heads occurs.

Binomial Distribution (5/11)

Example 2 (2/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

a) No head occurs.

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

Binomial Distribution (5/11)

Example 2 (3/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

a) No head occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$f(0) = \binom{10}{0} (0.5)^0 (0.5)^{10-0}$$

Binomial Distribution (5/11)

Example 2 (4/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

Binomial Distribution (5/11)

Example 2 (4/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$P(X \geq 2)$

Binomial Distribution (5/11)

Example 2 (4/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$P(X \geq 2) = f(2) + f(3) + \cdots + f(10)$$

Binomial Distribution (5/11)

Example 2 (4/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

b) At least 2 heads occurs.

$$f(x) = \binom{10}{x} (0.5)^x (0.5)^{10-x}$$

$$P(X \geq 2) = \cancel{f(2) + f(3) + \cdots + f(10)}$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [f(1) + f(0)]$$

Binomial Distribution (5/11)

Example 2 (5/5):

$X \sim \text{Bin}(10, 0.5)$

Flip a coin 10 times, what is the probability that:

b) At least 2 heads occurs.

$$f(0) = \binom{10}{0}(0.5)^0(0.5)^{10-0}$$

$$f(1) = \binom{10}{1}(0.5)^1(0.5)^{10-1}$$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [f(1) + f(0)]$$

Binomial Distribution (6/11)

Mean and Variance $b(n, p)$:

If X is a binomial random variable with parameters p and n

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/11)

Example 3 (1/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

Binomial Distribution (7/11)

Example 3 (2/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (7/11)

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Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

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- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/11)

Example 3 (4/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$
$$= 10 \times 0.5 = 5$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (7/11)

Example 3 (5/5):

Flip a coin 10 times. Let X = number of heads obtained, what are the expected value and the variance of X?

- Bernoulli trial → Flip a coin
- Number of trials ($n = 10$)
- Success → Head occur ($p = 0.5$)
- Let X = number of heads obtained
  - Values $x = 0, 1, 2, \dots, 10$

$$\mu = E(X) = np$$
$$= 10 \times 0.5 = 5$$

$$\sigma^2 = V(X) = np(1-p)$$
$$= 10 \times 0.5 \times 0.5 = 2.5$$

Binomial Distribution (8/11)

Example 4 (1/5):

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

Binomial Distribution (8/11)

Example 4 (2/5):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

Binomial Distribution (8/11)

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$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

Binomial Distribution (8/11)

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$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

Binomial Distribution (8/11)

Example 4 (4/5):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

(a) exactly 5 survive

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

$$f(5) = \binom{15}{5} (0.4)^5 (0.6)^{15-5}$$

$$= 0.1859$$

Binomial Distribution (8/11)

Example 4 (5/5):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive, and (b) at least 3 survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

(b) at least 3 survive

$$f(x) = \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

$$P(X \geq 3) = 1 - P(X < 3)$$
$$= 1 - [f(2) + f(1) + f(0)]$$

Binomial Distribution (9/11)

Example 5 (1/4):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

Binomial Distribution (9/11)

Example 5 (2/4):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

$$\mu = E(X) = np$$

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (9/11)

Example 5 (3/4):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\mu = E(X) = np$$
$$= 15 \times 0.4 = 6$$

$$\sigma^2 = V(X) = np(1-p)$$

Binomial Distribution (9/11)

Example 5 (4/4):

$X \sim \text{Bin}(15, 0.4)$

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what are the expected value and the variance of the survive?

- Bernoulli trial → Patient
- Number of trials ($n = 15$)
- Success → Patient recover ($p = 0.4$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 15$

$$\mu = E(X) = np$$
$$= 15 \times 0.4 = 6$$

$$\sigma^2 = V(X) = np(1-p)$$
$$= 15 \times 0.4 \times 0.6$$
$$= 3.6$$

Binomial Distribution (10/11)

Example 6 (1/4):

The probability that a patient recovers from a COVID-19 virus is 0.8. What is the probability that at least 3 of the next 10 patients having this virus will survive?

Binomial Distribution (10/11)

Example 6 (2/4):

$X \sim \text{Bin}(10, 0.8)$

The probability that a patient recovers from a COVID-19 virus is 0.8. What is the probability that at least 3 of the next 10 patients having this virus will survive?

- Bernoulli trial → Patient
- Number of trials ($n = 10$)
- Success → Patient recover ($p = 0.8$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (10/11)

Example 6 (2/4):

$X \sim \text{Bin}(10, 0.8)$

The probability that a patient recovers from a COVID-19 virus is 0.8. What is the probability that at least 3 of the next 10 patients having this virus will survive?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.8)^x (0.2)^{10-x}$$

- Bernoulli trial → Patient
- Number of trials ($n = 10$)
- Success → Patient recover ($p = 0.8$)
- Let X = the number of people who survive
  - Values $x = 0, 1, 2, \dots, 10$

Binomial Distribution (10/11)

Example 6 (3/4):

$X \sim \text{Bin}(10, 0.8)$

The probability that a patient recovers from a COVID-19 virus is 0.8. What is the probability that **at least 3** of the next 10 patients having this virus will survive?

$$P(X \geq 3) = 1 - P(X \leq 2)$$
$$= 1 - [f(2) + f(1) + f(0)]$$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.8)^x (0.2)^{10-x}$$

Binomial Distribution (10/11)

Example 6 (4/4):

$X \sim \text{Bin}(10, 0.8)$

The probability that a patient recovers from a COVID-19 virus is 0.8. What is the probability that **at least 3** of the next 10 patients having this virus will survive?

$$P(X \geq 3) = 1 - P(X \leq 2)$$

You can find $P(X \leq 2)$ from the **Table**. $P(X \leq 2) = 0.0001$

$$\therefore P(X \geq 3) = 1 - 0.0001$$

$$= 0.9999$$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{10}{x} (0.8)^x (0.2)^{10-x}$$

Binomial Distribution (11/11)

Example 7 (1/4):

A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state?

Binomial Distribution (11/11)

Example 7 (2/4):

A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state?

- Bernoulli trial → Vehicle
- Number of trials ($n = 9$)
- Success → vehicle is from out of state ($p = 0.25$)
- Let X = the number of vehicles are from out of state
  - Values $x = 0, 1, 2, \dots, 9$

Binomial Distribution (11/11)

Example 7 (3/4):

$X \sim \text{Bin}(9, 0.25)$

A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state?

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x) = \binom{9}{x} (0.25)^x (0.75)^{9-x}$$

- Bernoulli trial → Vehicle
- Number of trials ($n = 9$)
- Success → vehicle is from out of state ($p = 0.25$)
- Let X = the number of vehicles are from out of state
  - Values $x = 0, 1, 2, \ldots, 9$

Binomial Distribution (11/11)

Example 7 (4/4):

$X \sim \text{Bin}(9, 0.25)$

A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that **fewer than 4** of the next 9 vehicles are from out of state?

$P(X < 4) = P(X \leq 3)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$P(X \leq 3)$, $n = 9$, $p = 0.25$

You can find $P(X \leq 3)$ from the **Table**.

$P(X \leq 3) = 0.8343$

$$f(x) = \binom{9}{x} (0.25)^x (0.75)^{9-x}$$

Multinomial Distribution (1/5)

Multinomial Experiments:

The binomial experiment becomes a multinomial experiment if we let each trial have **more than two** possible outcomes.

Multinomial Distribution (2/5)

Multinomial Distribution:

If a given trial can result in the k outcomes $E_1, E_2, \ldots, E_k$ with probabilities $p_1, p_2, \ldots, p_k$, then the probability distribution of the random variables $X_1, X_2, \ldots, X_k$, representing the number of occurrences for $E_1, E_2, \ldots, E_k$ in n independent trials, is

$$f(x_1, x_2, \ldots x_k) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k},$$

with

$$\sum_{i=1}^{k} x_i = n \quad \text{and} \quad \sum_{i=1}^{k} P_i = 1$$

Multinomial Distribution (3/5)

Multinomial Distribution:

If a given trial can result in the k outcomes $E_1, E_2, \ldots, E_k$ with probabilities $p_1, p_2, \ldots, p_k$, then the probability distribution of the random variables $X_1, X_2, \ldots, X_k$, representing the number of occurrences for $E_1, E_2, \ldots, E_k$ in n independent trials, is

$$f(x_1, x_2, \ldots x_k) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k},$$

$$= \frac{n!}{x_1!\, x_2! \cdots x_k!}$$

Multinomial Distribution (4/5)

Multinomial Distribution – Example 1 (1/5):

The complexity of arrivals and departures of planes at an airport is such that computer simulation is often used to model the "ideal" conditions. For a certain airport with three runways, it is known that in the ideal setting the following are the probabilities that the individual runways are accessed by a randomly arriving commercial jet:

Runway1: $p_1 = 2/9$,
Runway2: $p_2 = 1/6$,
Runway3: $p_3 = 11/18$.

Multinomial Distribution (4/5)

Multinomial Distribution – Example 1 (2/5):

The individual runways are accessed by a randomly arriving commercial jet:

Runway1: $p_1 = 2/9$,
Runway2: $p_2 = 1/6$,
Runway3: $p_3 = 11/18$.

What is the probability that 6 randomly arriving airplanes are distributed in the following fashion?

Runway1: 2 airplanes,
Runway2: 1 airplane,
Runway3: 3 airplanes.

Multinomial Distribution (4/5)

Multinomial Distribution – Example 1 (3/5):

The individual runways are accessed by a randomly arriving commercial jet:

Runway1: $p_1 = 2/9$,
Runway2: $p_2 = 1/6$,
Runway3: $p_3 = 11/18$.

What is the probability that 6 randomly arriving airplanes are distributed in the following fashion?

Runway1: 2 airplanes, $x_1 = 2$
Runway2: 1 airplane, $x_2 = 1$
Runway3: 3 airplanes. $x_3 = 3$

Multinomial Distribution (4/5)

Multinomial Distribution – Example 1 (4/5):

The individual runways are accessed by a randomly arriving commercial jet:

Runway1: $p_1 = 2/9$,
Runway2: $p_2 = 1/6$,
Runway3: $p_3 = 11/18$.

$$f(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

$$f(2,1,3) = \binom{6}{2,1,3} \left(\frac{2}{9}\right)^2 \left(\frac{1}{6}\right)^1 \left(\frac{11}{18}\right)^3$$

What is the probability that 6 randomly arriving airplanes are distributed in the following fashion?

Runway1: 2 airplanes, $x_1 = 2$
Runway2: 1 airplane, $x_2 = 1$
Runway3: 3 airplanes. $x_3 = 3$

Multinomial Distribution (4/5)

Multinomial Distribution – Example 1 (5/5):

$$f(2,1,3) = \binom{6}{2,1,3}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3$$

$$= \frac{6!}{2!\,1!\,3!}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3 = 0.1127$$

Multinomial Distribution (5/5)

Multinomial Distribution – Example 2 (1/4):

The probabilities are 0.4, 0.2, 0.3, and 0.1, respectively, that a delegate to a certain convention arrived by air, bus, automobile, or train. What is the probability that among 9 delegates randomly selected at this convention, 3 arrived by air, 3 arrived by bus, 1 arrived by automobile, and 2 arrived by train.

Multinomial Distribution (5/5)

Multinomial Distribution – Example 2 (2/4):

The probabilities are 0.4, 0.2, 0.3, and 0.1, respectively, that a delegate to a certain convention arrived by air, bus, automobile, or train. What is the probability that among 9 delegates randomly selected at this convention, 3 arrived by air, 3 arrived by bus, 1 arrived by automobile, and 2 arrived by train.

$$p_1 = 0.4$$
$$p_2 = 0.2$$
$$p_3 = 0.3$$
$$p_4 = 0.1$$

$$x_1 = 3$$
$$x_2 = 3$$
$$x_3 = 1$$
$$x_4 = 2$$

Multinomial Distribution (5/5)

Multinomial Distribution – Example 2 (3/4):

$$f(x_1, x_2, x_3, x_4) = \binom{n}{x_1, x_2, x_3, x_4} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

$$f(3, 3, 1, 2) = \binom{9}{3, 3, 1, 2} (0.4)^3 (0.2)^3 (0.3)^1 (0.1)^2$$

$$p_1 = 0.4$$
$$p_2 = 0.2$$
$$p_3 = 0.3$$
$$p_4 = 0.1$$

$$x_1 = 3$$
$$x_2 = 3$$
$$x_3 = 1$$
$$x_4 = 2$$

Multinomial Distribution (5/5)

Multinomial Distribution – Example 2 (4/4):

$$f(3,3,1,2) = \binom{9}{3,3,1,2}(0.4)^3(0.2)^3(0.3)^1(0.1)^2$$

$$= \frac{9!}{3!\,3!\,1!\,2!}(0.4)^3(0.2)^3(0.3)^1(0.1)^2 = 0.0077$$

Negative Binomial (1/3)

Negative Binomial Distribution:

$X \sim nb(k, p)$

If repeated independent trials can result in a success with probability p, then the probability distribution of the random variable X, the number of the trial on which the kth success occurs, is

$$f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

with

$$k = 1,2, \ldots \quad \text{and} \quad x = k, k+1, k+2, \ldots$$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (1/8):

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

What is the probability that team A will win the series in 6 games?

Negative Binomial (2/3)

Negative Binomial Distribution – Example (2/8):

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that **team A has probability 0.55 of winning a game over team B**.

What is the probability that team A will win the series in 6 games?

$p = 0.55$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (3/8):

In an NBA (National Basketball Association) championship series, the team that wins **four games** out of seven is the winner. Suppose that teams A and B face each other in the championship games and that **team A has probability 0.55 of winning a game over team B**.

What is the probability that team A will win the series in **6 games**?

$p = 0.55, \quad x = 6, \quad k = 4$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (4/8):

What is the probability that team A will win the series in **6 games**?

$$p = 0.55, \quad x = 6, \quad k = 4$$

$$f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

$$f(6) = \binom{5}{3} (0.55)^4 (1-0.55)^{6-4} = 0.1853$$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (5/8):

What is the probability that team A will win the series?

Negative Binomial (2/3)

Negative Binomial Distribution – Example (6/8):

What is the probability that team A will win the series?

$p = 0.55,$

1) $x = 4, \quad k = 4$
2) $x = 5, \quad k = 4$
3) $x = 6, \quad k = 4$
4) $x = 7, \quad k = 4$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (7/8):

What is the probability that team A will win the series?

$p = 0.55,$

1) $x = 4, \quad k = 4$
2) $x = 5, \quad k = 4$
3) $x = 6, \quad k = 4$
4) $x = 7, \quad k = 4$

$f(4) + f(5) + f(6) + f(7)$

Negative Binomial (2/3)

Negative Binomial Distribution – Example (8/8):

What is the probability that team A will win the series?

$p = 0.55,$

1) $x = 4, \quad k = 4$
2) $x = 5, \quad k = 4$
3) $x = 6, \quad k = 4$
4) $x = 7, \quad k = 4$

$f(4) + f(5) + f(6) + f(7) = 0.6083$

$$f(4) = \binom{3}{3}(0.55)^4(1-0.55)^{4-4} = 0.0915$$

$$f(5) = \binom{4}{3}(0.55)^4(1-0.55)^{5-4} = 0.1647$$

$$f(6) = \binom{5}{3}(0.55)^4(1-0.55)^{6-4} = 0.1853$$

$$f(7) = \binom{6}{3}(0.55)^4(1-0.55)^{7-4} = 0.1668$$

Negative Binomial (3/3)

Negative Binomial Distribution: (Mean and Variance)

If X is a negative binomial random variable with parameters p and k,

$$\mu = E(X) = k/p \text{ and}$$

$$\sigma^2 = V(X) = k(1-p)/p^2$$

Geometric Distribution (1/3)

Geometric Distribution:

$X \sim g(p)$

If repeated independent trials can result in a success with probability p, then the probability distribution of the random variable X, the number of the trial on which the **first** success occurs, is

$$f(x) = p(1-p)^{x-1}$$

with

$$x = 1, 2, 3, \ldots$$

Geometric Distribution (2/3)

Geometric Distribution – Example (1/5):

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Geometric Distribution (2/3)

Geometric Distribution – Example (2/5):

For a certain manufacturing process, it is known that, on the average, **1 in every 100** items is defective. What is the probability that the fifth item inspected is the first defective item found?

$p = 0.01$

Geometric Distribution (2/3)

Geometric Distribution – Example (3/5):

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the **fifth item inspected** is the **first** defective item found?

$p = 0.01, \quad x = 5, \quad k = 1$

Geometric Distribution (2/3)

Geometric Distribution – Example (4/5):

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

$p = 0.01, \quad x = 5, \quad k = 1$

$f(x) = p(1-p)^{x-1}$

Geometric Distribution – Example (5/5):

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

$$p = 0.01, \quad x = 5, \quad k = 1$$

$$f(x) = p(1-p)^{x-1}$$

$$f(5) = (0.01)(1-0.01)^{5-1} = 0.0096$$

Geometric Distribution (3/3)

Geometric Distribution: (Mean and Variance)

If X is a geometric random variable with parameter p,

$$\mu = E(X) = 1/p \text{ and}$$

$$\sigma^2 = V(X) = (1-p)/p^2$$

Poisson Distribution (1/7)

Poisson Distribution:

$X \sim p(\lambda t)$ | $X \sim \text{Poisson}(\lambda t)$

The mean number of outcomes is computed from $\mu = \lambda t$, where t is the specific "time," "distance," "area," or "volume" of interest. Since the probabilities depend on λ, the rate of occurrence of outcomes.

$$f(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

with

$$\lambda > 0 \text{ and } \quad x = 0, 1, 2, \ldots$$

Poisson Distribution (2/7)

Poisson Distribution – Example 1 (1/4):

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Poisson Distribution (2/7)

Poisson Distribution – Example 1 (2/4):

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

$$\lambda t = 4$$
$$x = 6$$

Poisson Distribution (2/7)

Poisson Distribution – Example 1 (3/4):

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

$$\lambda t = 4$$
$$x = 6$$

$$f(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

Poisson Distribution – Example 1 (4/4):

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

$$\lambda t = 4$$
$$x = 6$$

$$f(6) = \frac{e^{-4}(4)^6}{6!} = 0.1042$$

Poisson Distribution (3/7)

Poisson Distribution:

$X \sim p(\lambda)$ | $X \sim \text{Poisson}(\lambda)$

$$f(x) = \frac{e^{-\lambda}(\lambda)^x}{x!}$$

with

$$\lambda > 0 \text{ and } \quad x = 0, 1, 2, \ldots$$

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (1/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (2/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

$$\lambda = 10,\ x > 15$$

$$f(x) = \frac{e^{-\lambda}(\lambda)^x}{x!}$$

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (3/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

$$\lambda = 10,\ x > 15$$

$$f(X > 15) = 1 - f(X \leq 15)$$

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (4/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

$$\lambda = 10,\ x > 15$$

$$f(X > 15) = 1 - f(X \leq 15)$$

$$f(X = 0)$$
$$f(X = 1)$$
$$f(X = 2)$$
$$\dots$$
$$f(X = 15)$$

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (5/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

$$\lambda = 10,\ x > 15$$

$$f(X > 15) = 1 - f(X \leq 15)$$

From the Table
$= 0.9513$

Poisson Distribution (4/7)

Poisson Distribution – Example 2 (6/6):

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

$$\lambda = 10,\ x > 15$$

$$f(X > 15) = 1 - 0.9513 = 0.0487$$

Poisson Distribution (5/7)

Poisson Distribution: (Mean and Variance)

Both the mean and the variance of the Poisson distribution are

$$E(X) = V(X) = \lambda t$$

Poisson Distribution (6/7)

Poisson Distribution: (From Binomial)

Let X be a binomial random variable with probability distribution $b(n, p)$.

When $n \to \infty$, $p \to 0$,

and $np \xrightarrow{n\to\infty} \mu$ remains constant,

$b(n, p) \xrightarrow{n\to\infty} p(\mu)$.

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (1/6)

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (2/6)

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, **1 in every 1000** of these items produced has one or more bubbles. What is the probability that a random **sample of 8000** will yield **fewer than 7** items possessing bubbles?

$p = 0.001,$ and $n = 8000$

$f(x < 7)$ from Binomial

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (3/6)

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, **1 in every 1000** of these items produced has one or more bubbles. What is the probability that a random **sample of 8000** will yield **fewer than 7** items possessing bubbles?

$p = 0.001,$ and $n = 8000$

$f(x < 7) = f(x \leq 6)$ from Binomial

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (4/6)

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

$p = 0.001,$ and $n = 8000$

$$f(x < 7) = f(x \leq 6) = \sum_{x=0}^{6} Bin(x; n, p)$$

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (5/6)

$p = 0.001,$ and $n = 8000$

$$f(x < 7) = f(x \leq 6) = \sum_{x=0}^{6} Bin(x; 8000, 0.001)$$

Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$$\mu = np = 8$$

$$\sum_{x=0}^{6} p(x; 8) = 0.3134$$

From the Table

Poisson Distribution (7/7)

Poisson Distribution: (From Binomial): Example (6/6)

$p = 0.001,$ and $n = 8000$

$$f(x < 7) = f(x \leq 6) = \sum_{x=0}^{6} Bin(x; 8000, 0.001) \approx 0.3134$$

Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$$\mu = np = 8$$

$$\sum_{x=0}^{6} p(x; 8) = 0.3134$$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #10: https://www.youtube.com/watch?v=bRK10hfwtoY&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=11

Thank You

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